# 2022

# MATHEMATICS

**Full Marks : 100**

**Pass Marks : 33**

**Time : Three hours**

*Attempt all Questions.*

*The figures in the right margin indicate full marks for the questions.*

*For Question Nos.* ***1 – 4,*** *write the letter associated with the correct answer.*

1. If $\sin^{-1} x = -\frac{\pi}{3}, x \in [-1, 1]$, then the value of $\cos^{-1} x$ is 1

A. $\frac{\pi}{3}$

B. $\frac{\pi}{6}$

C. $\frac{2\pi}{3}$

D. $\frac{5\pi}{6}$

2. The integral $\int \frac{dx}{\sqrt{x^2 + a^2}}$ equals 1

A. $\frac{1}{a} \tan^{-1} \frac{x}{a} + C$

B. $\log\left| x + \sqrt{x^2 + a^2} \right| + C$

C. $\sin^{-1} \frac{x}{a} + C$

D. $\log\left| x - \sqrt{x^2 + a^2} \right| + C$

3. A homogenous differential equation of the form $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$ can be solved by making the substitution 1

A. $y = vx$

B. $v = xy$

C. $x = vy$

D. $y = v$

4. The projection of the vector $\hat{i} - \hat{j}$ on the vector $\hat{i} + \hat{j}$ is 1

*A.* 0

*B.* 1

*C.* $\frac{1}{\sqrt{2}}$

*D.* $\frac{1}{2}$

5. Define a binary operation on a set. 1

6. What is the range of the principal value branch of the function $\sec^{-1}$? 1

7. Find the value of $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$. 1

8. Find $AB$ if $A = \begin{bmatrix} 2 & -3 \\ 4 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 1 & 3 \\ 2 & -4 & 1 \end{bmatrix}$. 1

9. For what value of $k$ is the function $f$ defined by

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$$

continuous at $x = 0$ ? 1

10. Find the second order derivative of the function $x.\cos x$. 1

11. Find $\int e^x (\sin x + \cos x)\, dx$. 1

12. Form the differential equation representing the family of curves $y = mx$, $m$ being arbitrary constant. 1

13. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2 respectively having $\vec{a}.\vec{b} = \sqrt{6}$. 1

14. Find the direction cosines of the $x$ - axis. 1

15. Prove that $2\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{7} = \tan^{-1}\dfrac{31}{17}$. 2

16. Prove that the inverse of a square matrix, if it exists, is unique. 2

17. Find the equation of tangent to the curve given by $x = \cos t$, $y = \sin t$ at a point where $t = \dfrac{\pi}{4}$. 2

18. Find the integral $\int \sin^3 x \cos^3 x\, dx$. 2

19. Write the steps involved to solve a first order linear differential equation. 2

20. If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, then show that 2

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

21. Show that the lines $\frac{x+3}{-3}=\frac{y-1}{1}=\frac{z-5}{5}$ and $\frac{x+1}{-1}=\frac{y-2}{2}=\frac{z-5}{5}$ are coplanar. 2

22. Find the mean of the number obtained on a throw of an unbiased die. 2

23. A person buys a lottery ticket in 50 lotteries, in each of which his chance of winning a prize is $\frac{1}{100}$. What is the probability that he will win a prize at least once? 2

24. Show that the relation R in the set A of all books in a library of a college, given by R={ $(x,y)$ : $x$ and $y$ have same number of pages} is an equivalence relation. 4

***OR***

Show that the function $f: R \to R$ given by $f(x) = 5x + 4$ is invertible. Find the inverse of $f$.

25. Express the matrix $A = \begin{vmatrix}3 & 3 & -1\\ -2 & -2 & 1\\ -4 & -5 & 2\end{vmatrix}$ as the sum of a symmetric and a skew symmetric matrix. 4

26. If $u, v$ and $w$ are functions of $x$, then show that

$$\frac{d}{dx}(u.v.w)=\frac{du}{dx}.v.w+u.\frac{dv}{dx}.w+u.v.\frac{dw}{dx}.$$

in two ways - first by repeated application of product rule, second by logarithmic differentiation. 4

*OR*

Prove that the greatest integer function $f$ given by $f(x) = [x]$, $0<x<3$ is not differentiable at $x = 1$ and $x = 2$.

27. Verify Mean Value Theorem, if $f(x) = x^2 - 4x - 3$ in the interval $[a, b]$, where $a = 1$ and $b = 4$. 4

28. Prove that $\int_{-a}^{a} f(x)dx = dx = 2\int_{0}^{a} f(x)dx$, if $f$ is an even function and 0, if $f$ is an odd function. 4

29. Using integration, find the area of the region bounded by the triangle whose vertices are (–1, 0),(1, 3) and (3, 2). 4

30. In a bank, principal increases continuously at the rate of 5% per year. An amount of Rs. 1000 is deposited with this bank. How much will it worth after 10 years ($e^{0.5} = 1.648$). 4

31. Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively, in the ratio 2 : 1 (i) internally (ii) externally. 4

32. Solve that following system of linear equations, using matrix method : 6

$$x - y + 2z = 7$$
$$3x + 4y - 5z = -5$$
$$2x - y + 3z = 12$$

33. Solve that the semi-vertical angle of the right circular cone of the maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$ . 6

***OR***

A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

34. Show that $\int \frac{(3\sin\theta - 2)\cos\theta}{5\cos^2\theta - 4\sin\theta}\, d\theta = 3\log(2 - \sin\theta) + \frac{4}{2 - \sin\theta} + C$. 6

***OR***

Show that $\int_0^{\pi} \log(1 + \cos x)\, dx = -\pi \log 2$.

35. Find the vector equation of a line through a given point and parallel to a given vector in the form $\vec{r} = \vec{a} + \lambda\vec{b}$. Also, derive the cartesian form $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$ a from the vector form. 6

***OR***

Find the vector equation of a plane in the normal form $\vec{r}.\hat{n} = d$. Also, derive the cartesian form $lx + my + nz = d$ from the vector form.

36. A direction wishes to mix together two kinds of food X and Y in such a way that the mixture contains at least 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C. The vitamin contents of one kg food is given below :

| Food | Vitamin A | Vitamin B | Vitamin C |
|---|---|---|---|
| X | 1 | 2 | 3 |
| Y | 2 | 2 | 1 |

One kg of food X costs Rs. 16 and one kg of food Y costs Rs. 20. Find the least cost of the mixture which will produce the required diet. 6

37. A manufacturer has three machine operators A,B and C. The first operator A produces 1% defective items, whereas the other two operators B and C produce 5% and 7% defective items respectively. A is on the job for 50% of the time, B is on the job for 30% of the time and C is on the job for 20% of the time. A defective item is produced, what is the probability that it was produced by A? 6

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